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COHOMOLOGY OF PERMUTATIVE CELLULAR AUTOMATA

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ABSTRACT

We examine $U(d)$ valued cocycles for a $\mathbb{Z}^{2+}$ action generated by a mixing, permutative cellular automaton and show that the set of Hölder continuous cocycles (for a given Hölder order) which are cohomologous to constant cocycles is both open and closed in the appropriate topology. A continuous dimension function with values in $\{0, 1, \ldots, d\}$ is defined on cocycles; a cocycle is cohomologous to a constant precisely when the value is d. When $d = 1$ (the abelian case) the first (essential) cohomology group is countable. If $U(1) \simeq$ circle is replaced by a finite subgroup, this cohomology group is finite.

Introduction

The study of endomorphisms of full shifts began with Hedlund's paper [2] and was followed by Coven and Paul's investigation of endomorphisms of shifts of finite type [1]. These papers prepared the way for the analysis of dynamical systems which have come to be known as cellular automata.

In this paper we shall understand a cellular automaton to be a $\mathbb{Z}^{2+}$ lattice dynamical system generated by a one-sided shift of finite type and an endomorphism of this shift. In fact we shall concentrate our attention on the case where the endomorphism is *permutative* in the sense introduced by Hedlund.

Our purpose is to understand certain rigidity properties associated with the cohomology of many $\mathbb{Z}^2$ (and $\mathbb{Z}^{2+}$) lattice dynamical systems. This paper is part of a general enquiry undertaken, for example, in [3], [10]. In the closely related papers by Schmidt and the author [6], [11], the underlying configuration spaces are algebraically defined and the conclusion drawn (with appropriate hypotheses)

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is that cocycles are cohomologous to algebraic cocycles. It is therefore natural to abandon the algebraic context (except, of course, as a rich source of important examples) and to find out what can be said of cocycles in a different setting. We have chosen to investigate the cohomology of permutative cellular automata, where all cocycles will be Hölder continuous.

Although we concentrate here on $\mathbb{Z}^{2+}$ actions our results will have a bearing on similarly defined $\mathbb{Z}^2$ actions, the latter being projective limits of the former. Of course, some work (not much) is required to show that this is the case.

Although we take a brief look at real (and $\mathbb{C}^d$) valued cocycles (which turn out to be cohomologous to constants) our main interest is in cocycles with values in a compact group, specifically $U(d)$. Such cocycles are naturally associated with group extensions of our $\mathbb{Z}^{2+}$ action. If the cocycle is a coboundary, the extended $\mathbb{Z}^{2+}$ action is a trivial extension in that it is essentially the direct product of the original action with the identity. Similarly, if the cocycle is cohomologous to a constant cocycle the extended $\mathbb{Z}^{2+}$ action is a direct product of the original action with translations in $U(d)$. It is reasonable, therefore, to regard such cocycles as essentially trivial.

It is known (cf. [6], [10]) that there are $\mathbb{Z}^{2+}$ actions where all (Hölder) circle valued ($d = 1$) cocycles are trivial in this sense. However, this is not always so.

For the space of Hölder cohomology classes modulo constants we define a dimension function $\dim_\theta$ with values in $(0, 1, \ldots, d)$ (here $0 < \theta < 1$ is the Hölder order) which turns out to be continuous, and $\dim_\theta = d$ precisely for the trivial class.

When we take $d = 1$ (circle valued cocycles) the above classes form an abelian group — the essential (first) cohomology group $H_0^1(K)$ — which is countable and discrete (and sometimes trivial).

When we further specialise to a finite subgroup G of K (the circle), $H_0^1(G)$ is a finite group.

The essential idea is to show that for mixing $\mathbb{Z}^{2+}$ actions generated by a permutative cellular automaton, the property of a Hölder cocycle being cohomologous to a constant cocycle is stable under perturbation. That is, a nearby cocycle shares the same property.

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1. Preliminaries

Let A be a $0-1$ aperiodic matrix of dimension k and let

$$X_A = \left\{ x \in \prod_{n=0}^{\infty} \{1, 2, \dots, k\} \colon A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}^+ \right\}.$$

With the topology inherited from the Tychonov topology on the direct product space, X_A is a compact zero-dimensional metrisable space which is invariant with respect to the shift σ given by

$$(\sigma x)_n = x_{n+1}.$$

The shift restricted to X_A is called a **shift of finite type** (defined by A).

An endomorphism of this shift is a continuous surjective map τ of X_A such that

$$\sigma\tau = \tau\sigma.$$

It is easy to see [2] that endomorphisms necessarily have the form

$$(\tau x)_n = \tau_0(x_n, x_{n+1}, \dots, x_{n+\ell}), \quad n \in \mathbb{Z}^+$$

where τ_0 is a map from the set of all (allowable) words of length $\ell + 1$ to the set of symbols $1, 2, \dots, k$. Such an endomorphism is called **permutative** if for each word $x_1, \dots, x_\ell$ the map

$$x_0 \to \tau_0(x_0, \dots, x_\ell)$$

is a bijection (from the set of predecessors of x_1 into the set of symbols). Equivalently, $\sigma x = \sigma y$ and $\tau x = \tau y$ imply $x = y$.

The $\mathbb{Z}^{2+}$ action on X_A generated by σ and an endomorphism τ is called a **cellular automaton**. We can display a point $x = \{x_{n,0}\} \in X_A$ either as an infinite sequence of symbols which is shifted to the left by σ (first symbol deleted) or as an array,

— — —	— — —	— — —	— — —
$x_{0,2}$	$x_{1,2}$	$x_{2,2}$	....
$x_{0,1}$	$x_{1,1}$	$x_{2,1}$	....
$x_{0,0}$	$x_{1,0}$	$x_{2,0}$	....

where $x_{m,n} = \tau_0(x_{m,n-1}, \dots, x_{m+\ell,n-1})$, in which case τ can be interpreted as the vertical shift, i.e. deletion of the bottom row, and σ is the horizontal shift to the left (deletion of the left column). Of course, each row is completely determined by its predecessor.

Thus a cellular automaton can be displayed as a $\mathbb{Z}^{2+}$ lattice dynamical system generated by the horizontal and vertical shifts. On the other hand, complete information is provided by σ and τ acting on our original space X_A.

In general it is a non-trivial problem to construct cellular automata: given τ_0, when is τ a surjection from X_A to itself? There is also the problem of deciding when τ is permutative. However, many examples are provided by an algebraic construction generalising Ledrappier's example [4].

2. Algebraic examples

Let p be a prime number and let d: $\mathbb{Z}^{2+} \to \mathbb{Z}/p$ have finite support

$$D = \{(m,n) \in \mathbb{Z}^{2+}\colon d(m,n) \not\equiv 0 \bmod \mathrm{p}\}$$

where $D + (m,n) \subset \mathbb{Z}^{2+}$ implies $(m,n) \in \mathbb{Z}^{2+}$.

Define

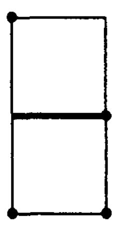
$$X_d = \Big\{x \in (\mathbb{Z}/p)^{\mathbb{Z}^{2+}}\colon \sum_{(m,n)\in\mathbb{Z}^{2+}} d(m+r,n+s)x(m,n) = 0 \text{ for all } (r,s) \in \mathbb{Z}^{2+}\Big\}.$$

Evidently X_d consists of points which are $\mathbb{Z}^{2+}$ arrays of symbols chosen from $\mathbb{Z}/p$ and is σ, τ invariant, where, as usual, σ, τ are the horizontal and vertical shifts; X_d is a compact metrisable abelian group and σ, τ are commuting surjective endomorphisms. Ledrappier's example is provided by d whose support D is $(0,0)$, $(0,1)$, $(1,0)$ and where d has value 1 on each of these points.

In other words X_d consists precisely of those points ($\mathbb{Z}^{2+}$ arrays) such that the sum of coordinates on D (or any translate of D) is zero mod p.

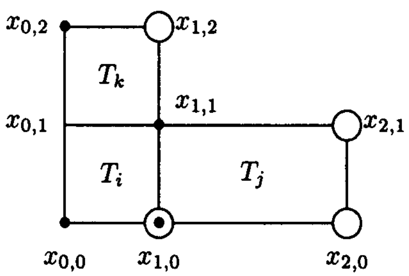
If D has a single point with maximum y coordinate and if this point lies on the y axis (here we are only giving *sufficient* conditions), then X_d can be represented as a cellular automaton.

An example will suffice. Let $D = (0,0),(1,0),(1,1),(0,2)$ and let d have value 1 on these points:



Let X be the space of all configurations of symbols from $\mathbb{Z}/p$ on the lattice $\mathbb{Z}^+ \times \{0,1\}$. It is easy to see that each point in X determines a configuration of symbols on the lattice $\mathbb{Z}^+ \times \{0,1,2\}$ if we require that the sum of d values over D (or its translates) should be zero mod p. And again the resulting configuration determines one on $\mathbb{Z}^+ \times \{0,1,2,3\}$ and so on. Thus a point of X determines a configuration on $\mathbb{Z}^+ \times \mathbb{Z}^+$.

The space X_d can be represented as the configuration space of a cellular automaton as follows. For symbols we take all unit square 'tiles' which consist of unit squares with elements from $\mathbb{Z}/p$ placed at vertices. There are p^4 such tiles, $T_1, T_2, \ldots, T_{p^4}$. The shift of finite type σ is defined by the matrix A where $A(T_i, T_j) = 1$ if and only if the tiles T_i and $T_j + (1,0)$ agree on their overlap. The function τ_0 is given by $T_k = \tau_0(T_i, T_j)$ when, in the following diagram:



we have $x_{0,0} + x_{1,0} + x_{1,1} + x_{0,2} = 0$ and $x_{1,0} + x_{2,0} + x_{2,1} + x_{1,2} = 0$.

The above example (like Ledrappier's example) is permutative. This follows from the fact that $(0,0) \in D$, for in this case $T_i \to \tau_0(T_i, T_j)$ is a bijection for each T_j, or in other words the tiles T_k and T_j determine T_i (i.e. $x_{0,0}$).

3. Cocycles of cellular automata

From here on we shall be concerned with a cellular automaton which, as we have said, is a $\mathbb{Z}^{2+}$ action generated by a shift of finite type σ (on $X = X_A$) and an endomorphism τ ($\sigma\tau = \tau\sigma$).

We shall also suppose that the automaton is permutative and *mixing*. This latter condition requires that for all non-empty open sets U, V

$$\sigma^{-m}\tau^{-n}U \cap V \neq \emptyset$$

whenever $m+n$ is large enough. For the algebraic examples the condition amounts to

$$\chi(\sigma^m\tau^n) = \chi,$$

where χ is a character implies that χ is trivial or $m = n = 0$. (See also [12] for a study of ergodic properties of permutative cellular automata.)

We shall investigate Hölder continuous cocycles with values in a compact Lie group or, equivalently, in $U(d)$, the group of d-dimensional unitary matrices.

If f is a continuous function on X with values in a Euclidean space ($\mathbb{C}^d$ or $\mathbb{R}$, for example) then the nth variation is defined as

$$\mathrm{var}_n f = \sup\{|f(x) - f(y)|: x_i = y_i, 0 \le i \le n\}$$

and we write $f \in F_\theta$ $\big(= F_\theta(\mathbb{C}^d), F_\theta(\mathbb{R}), F_\theta(U(d))\big)$ according to context, if

$$\mathrm{var}_n f \le K\theta^n, \quad n = 0, 1, \ldots$$

for some constant K. Here $0 < \theta < 1$ and $||$ denotes the Euclidean norm. The least such K is denoted $|f|_\theta$.

For such functions we say that f is Hölder of order θ. The spaces $F_\theta(\mathbb{C}^d), F_\theta(\mathbb{R})$ are Banach spaces when endowed with the norm

$$\|f\|_\theta = |f|_\infty + |f|_\theta.$$

An F_θ $\big(= F_\theta(\mathbb{C}^d), F_\theta(\mathbb{R})\big)$ **cocycle** is an F_θ pair (f, g) such that

$$f \circ \tau - f = g \circ \sigma - g.$$

If (f', g') is another F_θ cocycle, we say that it is **cohomologous** to the first if there exists an F_θ function h such that

$$h \circ \sigma + f = f' + h \quad \text{and} \quad h \circ \tau + g = g' + h.$$

When $(f', g') = (0, 0)$ is trivial (f, g) is called a **coboundary**.

In the same way an $F_\theta\,(U(d))$ **cocycle** is a pair (f, g) $\big($each belonging to $F_\theta\,(U(d))\big)$ such that

$$g \circ \sigma \cdot f = f \circ \tau \cdot g,$$

and (f, g), (f', g') are **cohomologous** if there exists $h \in F_\theta\,(U(d))$ such that

$$h \circ \sigma \cdot f = f' \cdot h \quad \text{and} \quad h \circ \tau \cdot g = g' \cdot h.$$

In case $(f', g') = (1, 1)$ is trivial, (f, g) is called a **coboundary**. We shall be particularly interested in the case where f', g' are constants (necessarily commuting) when we say that (f, g) is an **essential coboundary**.

For the case $d = 1$ ($U(1) = K$, the circle) there are examples of mixing permutative $\mathbb{Z}^{2+}$ actions where all $F_\theta(K)$ cocycles are essential coboundaries and, in contrast, there are examples which have $F_\theta(K)$ cocycles which are not essential coboundaries.

For example, all $F_\theta(K)$ cocycles for Ledrappier's $\mathbb{Z}^{2+}$ action are essential coboundaries whereas for the other example mentioned in Section 2, the algebraic cocycle (γ, η) given by homomorphisms into $\mathbb{Z}$mod2 where

$$\gamma(x) = x_{0,1}, \eta(x) = x_{0,0}$$

is not an essential coboundary. (These examples and many others are discussed in [11].)

However, it is interesting to note that for $d > 1$ there are $F_\theta\,(U(d))$ cocycles for Ledrappier's example which are not essential coboundaries.

Such an example was shown to me by Klaus Schmidt. There is a subgroup G of X which is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ and which is preserved by σ and τ. Moreover, $\tau = \sigma^2$ when restricted to G, and σ has order 3 on G, permuting all elements of G (other than the identity) cyclically. Thus σ, τ induce endomorphisms σ, τ on X/G. Since $X \simeq X/G \times G$ (as spaces) we can view σ, τ on X as a G extension of σ, τ on X/G — however, G does not commute with σ, τ. Instead we have a cocycle identity of the form

$$g(\sigma x)\tau\big(f(x)\big) = f(\tau x)\sigma\big(g(x)\big)$$

where $f, g\colon X/G \to G$.

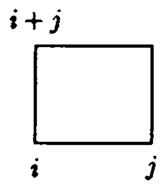
This equation can be made into a cocycle identity in the sense we have been using if we extend the value group G to the group A_4 (the alternating group), represented as the semi-direct product $G \times_\sigma \mathbb{Z}/3$ where

$$(k, m) \cdot (h, n) = (k + \sigma^m(h), m + n)\,.$$

The functions f, g can then be lifted to functions on X. This then gives an A_4 valued cocycle on X which is not cohomologous to a constant.

Alternatively we can directly construct f, g with values in A_4 (represented as 4×4 permutation matrices) as follows:

We define f, g as functions of one variable, i.e. $f(x) = f(x_{0,0}), g(x) = g(x_{0,0})$. And since we are considering Ledrappier's example we shall require (as indicated by the diagram)



$f(i)g(j) = g(i)f(i+j)$ for $i, j = 0, 1$. In other words

$$\begin{aligned} f(0)g(0) &= g(0)f(0), \\ f(0)g(1) &= g(0)f(1), \\ f(1)g(0) &= g(1)f(1), \\ f(1)g(1) &= g(1)f(0). \end{aligned}$$

When this is done (f^{-1}, g^{-1}) will be a cocycle. We define

$$f(0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad f(1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$g(0) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g(1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and one checks that the equations are satisfied.

It is clear that (f^{-1}, g^{-1}) is not an essential coboundary, for the equations

$$\begin{aligned} f^{-1}(0) &= h^{-1}(0)\alpha h(0) \\ &= h^{-1}(1)\alpha h(0), \\ f^{-1}(1) &= h^{-1}(0)\alpha h(1) \\ &= h^{-1}(1)\alpha h(1) \end{aligned}$$

would need to be satisfied, so that $h(0) = h(1), f(0) = f(1)$. However f is *not* constant.

It is well known (cf. [9], [7] for example) that if $f \in F_\theta(\mathbf{R})$ then the Ruelle–Perron–Frobenius operator

$$L_f \colon F_\theta(\mathbf{C}) \to F_\theta(\mathbf{C})$$

defined by

$$(L_f w)(x) = \sum_{\sigma y = x} (e^{f(y)} w(y))$$

has a unique maximum eigenvalue α (which is simple) and an eigenfunction h can be taken to be strictly positive. Moreover, the rest of the spectrum is contained in a disc of radius less than α.

Hence $L_f h = \alpha h$, and if we write $f' = f + \log h - \log h \circ \sigma - \log\alpha$ then $L_{f'} 1 = 1$, when we say that f' is **normalised.** If (f, g) is a real cocycle and f is normalised to f', then g can also be normalised to g' (where $L_{g'} 1 = 1$) in such a way that (f, g) is cohomologous to (f', g') plus a constant (pair) $(\log\alpha, \log\beta)$. We say the cocycle (f', g') is **normalised.**

THEOREM 3.1: *If σ, τ generate a permutative, mixing $\mathbb{Z}^{2+}$ action then every real F_θ cocycle is cohomologous to a constant.*

Proof: Let (f, g) be a normalised F_θ cocycle. We note that $L_f 1 = 1$ and (using the permutative condition) $L_f e^{-g} = e^{-g}$. Hence e^{-g} is constant, i.e. g is constant and, since $f\tau - f = g\sigma - g$, f is constant.

COROLLARY 3.2: *With the same hypothesis, σ is exactly k to 1 and τ is exactly ℓ to 1 for some k, ℓ.*

Proof: Normalise $(0, 0)$ to (f, g) where f, g are constant. Then

$$\sum_{\sigma y = x} e^{-f(y)} 1 = 1 = e^{-f(y)} (\text{card}\sigma^{-1} x)$$

and we see that $\text{card}\sigma^{-1} x$ is constant. In the same way $\text{card}\tau^{-1} x$ is constant.

COROLLARY 3.3: *With the same hypothesis every $\mathbb{C}^d$ valued F_θ cocycle is cohomologous to a constant.*

4. Cocycles with values in $U(d)$

Throughout this section, σ, τ generate a mixing, permutative $\mathbb{Z}^{2+}$ action. If (f, g) is an $F_\theta\,(U(d))$ cocycle we define Ruelle operators

$$L_f, L_g \colon F_\theta(\mathbb{C}^d) \to F_\theta(\mathbb{C}^d)$$

by incorporating the normalisation of the trivial real cocycle $(0,0)$ as follows:

$$(L_f w)(x) = \left(\sum_{\sigma y = x} f(y)w(y)\right) / \deg\sigma,$$

$$(L_g w)(x) = \left(\sum_{\tau y = x} g(y)w(y)\right) / \deg\tau,$$

where $\deg\sigma = \mathrm{card}\sigma^{-1}x$ (independent of x) and $\deg\tau = \mathrm{card}\tau^{-1}x$ (independent of x).

The following identities are easily checked:

$$L_f L_g = L_g L_f,$$

and if V_f, V_g are defined by $V_f w = f^{-1} w \circ \sigma, V_g w = g^{-1} w \circ \tau$ then

$$V_f V_g = V_g V_f,$$

$$L_f V_f = \mathrm{id}, \quad L_g V_g = \mathrm{id}.$$

Moreover, since σ, τ generate a permutative $\mathbb{Z}^{2+}$ action, then

$$L_f V_g = V_g L_f \quad \text{and} \quad L_g V_f = V_f L_g.$$

(These last equations are the reason for the permutativity assumption.)

Throughout this section we shall use $\langle , \rangle$ to denote the Euclidean inner product on $\mathbb{C}^d$ and we shall use $\langle\langle , \rangle\rangle$ to denote the inner product

$$\langle\langle v, w \rangle\rangle = \int \langle v, w \rangle dm$$

on $\mathbb{C}^d$ valued functions v, w, where m is the measure of maximal entropy for σ (which is also preserved by τ).

We shall need the following elementary lemma:

LEMMA 4.1: *If w_i: $X \to \mathbb{C}^d$ is F_θ for $i = 1, \dots, \ell$ where $\ell \leq d$ and $w_1(x), \dots, w_\ell(x)$ are orthonormal at each $x \in X$, then there exist $w_{\ell+1}, \dots, w_d$: $X \to \mathbb{C}^d$ which are F_θ and such that $w_1(x), \dots, w_d(x)$ are orthonormal at each $x \in X$.*

Proof: The proof is inductive so we need only produce $w_{\ell+1}$ when $\ell < d$. For an arbitrary $x_0 \in X$ choose $w'_{\ell+1}(x_0)$ of unit norm and orthogonal in

$\mathbb{C}^d$ to $w_1(x_0), \ldots, w_\ell(x_0)$. Define $w'_{\ell+1}(x) = w'_{\ell+1}(x_0)$ on a sufficiently small closed-open set containing x_0 so that $w'_{\ell+1}(x)$ is independent of $w_1(x), \ldots, w_\ell(x)$ in this neighbourhood. Define $w_{\ell+1}(x)$ in this neighbourhood by subtracting 'components' contributed by $w_1(x), \ldots, w_\ell(x)$ and then normalising. Thus $w_1(x), \ldots, w_\ell(x), w_{\ell+1}(x)$ are orthonormal for each x in the neighbourhood. $w_{\ell+1}$ is then defined on X by piecing together these local functions using compactness and zero dimensionality. It is easy to see that w_{l+1} is F_θ.

We shall use Pollicott's observation [8] regarding Browder's essential spectrum and Nussbaum's [5] formula for the essential spectral radius of a Banach space operator. Pollicott proved that the essential spectral radius of

$$L_f\colon F_\theta(\mathbb{C}^d) \to F_\theta(\mathbb{C}^d)$$

(when f is F_θ) is θ. (Actually this was proved for a complex valued f and $d = 1$, but the same proof works for $f \in F_\theta(U(d))$.)

As a consequence the eigenspaces corresponding to eigenvalues α with $|\alpha| > \theta$ are finite dimensional and there are only finitely many eigenvalues α for which $|\alpha| \geq \theta'$ when $\theta' > \theta$.

Thus the L_f invariant subspace corresponding to *all* eigenvalues $|\alpha| \geq \theta'$ is finite dimensional. Call this space $E_{\theta'}$ and equip it with the inner product $\langle\langle , \rangle\rangle$. Under the conditions of the next theorem we shall see that $\dim E_\theta (E_\theta = \bigcup_{\theta' > \theta} E_{\theta'})$ is at most d.

THEOREM 4.2: *If σ, τ generate a permutative, mixing $\mathbb{Z}^{2+}$ action and $(f,g) \in F_\theta(U(d))$ is a cocycle, then the restrictions to E_θ of L_f, L_g are finite dimensional unitary operators and $\dim E_\theta \leq d$. Furthermore, (f,g) is cohomologous to a cocycle (f', g') which is constant on a subspace of $\mathbb{C}^d$ of dimension $\dim E_\theta$. In particular, when $\dim E_\theta = d, (f,g)$ is cohomologous to a constant cocycle.*

Proof: Choose $1 > \theta' > \theta$, then $L_f\colon E_{\theta'} \to E_{\theta'}$ is a finite-dimensional operator and

$$E_{\theta'} = \bigoplus_{|\alpha| \geq \theta'} E_\alpha$$

where E_α: $\{w\colon (L_f - \alpha I)^n w = 0, \text{ some } n\}$. Since $V_g L_f = L_f V_g$ (by permutativity) and $L_g L_f = L_f L_g$ we see that $L_g, V_g, L_f\colon E_\alpha \to E_\alpha$ and hence $E_{\theta'} \to E_{\theta'}$. With respect to $\langle\langle . \rangle\rangle V_g$ is an isometry. Let E be the space of finite linear combinations of F_θ eigenfunctions for V_g so that $E \supset E_{\theta'}$.

If $V_g v = \alpha v, V_g w = \beta w$ then $\langle v, w\rangle \circ \tau = \alpha\bar{\beta}\langle v, w\rangle$, and since τ is mixing $\langle v(x), w(x)\rangle = 0$ for all $x \in X$ if $\alpha \neq \beta$. In any case $\langle v(x), w(x)\rangle$ is constant, i.e. for x_0 arbitrary $\langle v(x), w(x)\rangle = \langle v(x_0), w(x_0)\rangle$. Hence one sees that the map $E \to \mathbf{C}^d$, $v \to v(x_0)$ is an isometry, so E is finite dimensional with dimension no more than d, and V_g: $E \to E$ is a finite-dimensional isometry. It follows that V_f: $E \to E$ is a finite-dimensional isometry. Since E is spanned by eigenfunctions of V_f and $V_f w = \beta w$ implies $w = \beta L_f w, w \in W_{\theta'}$ we see that $E = E_{\theta'}$. This is true for all $\theta < \theta' < 1$ so $E = E_\theta$. In short

$$L_f, L_g, V_g, V_f\colon E_\theta \to E_\theta.$$

Let $w_1, \ldots, w_\ell$ be an orthonormal basis of eigenfunctions for V_f, V_g: $E_\theta \to E_\theta$:

$$V_f w_i = \alpha_i w_i \quad \text{and} \quad V_g w_i = \beta_i w_i$$

$(i = 1, \ldots, \ell)$ and note that $\langle w_i, w_j\rangle$ is constant and therefore zero when $i \neq j$, since σ is mixing. Hence $w_1(x), \ldots, w_\ell(x)$ are orthonormal at each $x \in X$. Using Lemma 4.1 we extend to $w_1, \ldots, w_\ell, \ldots, w_d$ orthonormal at each $x \in X$. (There is no claim that $w_{\ell+1}, \ldots, w_d$ are eigenfunctions.)

We define an $F_\theta\,(U(d))$ function W to have the $U(d)$ value $(w_1(x), \ldots, w_d(x))$ at each $x \in X$, so that $(f', g') = \big(W^{-1}(\sigma) f W, W^{-1}(\tau) g W\big)$ is a cohomologous cocycle, and in view of the equations

$$w_i(\sigma x) = \alpha_i f(x) w_i(x) \quad \text{and} \quad w_i(\tau x) = \beta_i g(x) w_i(x)$$

$(i = 1, \ldots, \ell) f', g'$ leave the subspace of $\mathbf{C}^d$ spanned by the first ℓ unit vectors invariant and the operators $f'(x), g'(x)$ are each independent of $x \in X$ on this subspace, i.e. f', g' are constant on this subspace.

When $\dim E_\theta = d$, this means that f', g' are both constant (diagonal) unitary matrices, so (f, g) is cohomologous to a constant (pair of commuting matrices).

We have just seen that if the $F_\theta\,(U(d))$ cocycle (f, g) is such that $\dim E_\theta = d$, then (f, g) is cohomologous to a constant cocycle (α, β) (i.e. $\alpha, \beta \in U(d)$ and $\alpha\beta = \beta\alpha$). In fact we can take α, β to be diagonal unitary matrices by choosing a suitable orthonormal basis for $\mathbf{C}^d$.

The converse is also true, for if $f = h^{-1}\sigma\alpha h$ and $g = h^{-1}\tau\beta h$ where (say) α, β are diagonal unitary matrices such that $\alpha\beta = \beta\alpha$ and $h \in F_\theta\,(U(d))$, then

$$L_f w_i = h^{-1} L_1(\alpha h w_i) = \alpha_i w_i, \quad i = 1, \ldots, d$$

if we choose $hw_i = \delta_i$, the ith unit vector in $\mathbf{C}^d$. Similarly

$$L_g w_i = \beta_i w_i, \quad i = 1, \ldots, d.$$

Here,

$$\alpha = \begin{pmatrix} \alpha_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \alpha_d \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \beta_d \end{pmatrix}.$$

Thus $\dim E_\theta = d$.

In the A_4 valued cocyles of Section 3 we represented $f(i), g(i)$ by

$$f(0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad f(1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$g(0) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g(1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and noted that

$$f(0)g(0) = g(0)f(0) = \mathrm{id},$$

$$f(0)g(1) = g(0)f(1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$f(1)g(0) = g(1)f(1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$f(1)g(1) = g(1)f(0) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

so that $\left(f^{-1}, g^{-1}\right)$ is a cocycle. Moreover $f, g, \in F_\theta\left(U(4)\right)$ for all $\theta > 0$. It is not difficult to show that if $L_{f^{-1}} w = \alpha w$ ($|\alpha| > 0$) then w is a function of one variable and there is only one solution: $\alpha = 1$, w constant. Moreover, the operator $L_{f^{-1}}$ active on the 8-dimensional space of functions of 1 variable has the characteristic polynomial $(\lambda - 1)\lambda^7$. This means that $\dim E_\theta = 1 < d = 4$.

Theorem 4.2 suggests the definition: $\dim_\theta(f,g) = \dim E_\theta = \dim_\theta(f)$.

THEOREM 4.3: *If σ, τ generate a permutative and mixing $\mathbb{Z}^{2+}$ action, then with respect to the relative topology induced by $F_\theta\,(U(d)) \times F_\theta\,(U(d))$, cocycles which are cohomologous to constants form an open and closed subset, i.e. $\{(f,g)\colon \dim_\theta(f,g) = d\}$ is open and closed.*

Proof: First we remark that cocycles form a closed subset of $F_\theta\,(U(d)) \times F_\theta\,(U(d))$, as is easily checked. Let (f,g) be cohomologous to a constant; then if (f',g') is a sufficiently near cocycle it follows that $L_{f'}$ is F_θ close to L_f and, by the perturbation theory of linear operators, it follows that $\dim_\theta(f') = \dim_\theta(f) = d$. (Here we make use of the fact that $E_\theta = E_{\theta'}$ for $1 \geq \theta' > \theta$, in our situation.) By the previous theorem it follows that (f',g') is cohomologous to a constant. Hence the set of cocycles cohomologous to constants is open.

Suppose $(f_n,g_n),(f,g)$ are $F_\theta\,(U(d))$ cocycles such that $f_n \to f$ and $g_n \to g$ in the $F_\theta\,(U(d))$ topology and $f_n = h_n^{-1} \circ \sigma\alpha_n h_n, g_n = h_n^{-1} \circ \tau\beta_n h_n$ where $\alpha_n\beta_n = \beta_n\alpha_n$ and α_n, β_n are diagonal unitary matrices and $h_n \in F_\theta\,(U(d))$. Then

$$L_{f_n} w_n^i = \alpha_n^i w_n^i$$

with

$$w_n^i = h_n^{-1}\delta_i \quad \text{and} \quad \alpha_n = \begin{pmatrix} \alpha_n^1 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \alpha_n^d \end{pmatrix}.$$

Using the *uniform* equicontinuity of the sequence $f_1, f_2, \ldots$ (implied by the F_θ topology) (cf. [7]) we can choose a uniformly convergent subsequence to get

$$L_f w^i = \alpha^i w^i, \quad L_g w^i = \beta^i w^i.$$

One can check that like $w_n^1, \ldots, w_n^d, w^1, \ldots, w^d$ are orthonormal at each point x. Furthermore, the following argument will show that $w^1, \ldots, w^d$ are not just continuous functions but are members of $F_\theta(\mathbb{C}^d)$.

We form the matrix $W = (w^1, \ldots, w^d)$, which is unitary at each point x and note that

$$W = f^{-1} W \circ \sigma\alpha \quad \text{where} \alpha = \begin{pmatrix} \alpha^1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \alpha^d \end{pmatrix}.$$

We choose $v_i \in F_\theta(\mathbb{C}^d)$ so that $W^{-1}v_i$ is close to the ith unit vector δ_i and see that

$$L_f W = W\alpha L_1.$$

Hence $L_f^n v_i = W\alpha L_1^n W^{-1} v_i$ and, taking a uniformly convergent subsequence, we get

$$F_\theta \ni v_i^* = W\alpha^* \int (W^{-1}v_i) dm$$

where the latter integral is close to $\delta_i, i = 1, \ldots, d$. Thus the constant matrix $(\int W^{-1} v_i dm \cdots \int W^{-1} v_d dm)$ is invertible and $W \in F_\theta$.

Having formed the matrix $W = (w^1, \ldots, w^d) \in F_\theta\,(U(d))$ we define its inverse h to get

$$f = h^{-1}\sigma\alpha h \quad \text{where} \alpha = \begin{pmatrix} \alpha^1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \alpha^d \end{pmatrix}$$

and

$$g = h^{-1}\tau\beta h \quad \text{where} \beta = \begin{pmatrix} \beta^1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \beta^d \end{pmatrix}.$$

To arrive at the last assertion one can infer it from the proof of Theorem or one can use the cocycle properly as follows: Since $g\sigma f = f\tau g$ we have

$$g\sigma(h^{-1}\sigma\alpha h) = \left(h^{-1}(\sigma\tau)\alpha h\tau\right) g$$

and therefore

$$F \circ \sigma = \alpha F \alpha^{-1},$$

where $F = h\tau g h^{-1}$. If we compare both sides we have $F_{i,j} \circ \sigma$ is a constant multiple of $F_{i,j}$ for each $i, j = 1, \ldots, d$, and since σ is mixing this implies each

$F_{i,j}$ is constant. Thus $F = \beta$, a constant, i.e. $g = h^{-1}\tau\beta h$. Of course, $\beta\alpha = \alpha\beta$ so β can be taken as a diagonal unitary matrix. Hence (f, g) is cohomologous to a constant.

COROLLARY 4.3: *Under the same hypotheses as the theorem, the function* $\dim_\theta$ *which maps* $F_\theta\,(U(d))$ *cocycles into* $\{0, 1, \ldots, d\}$ *is continuous.*

Proof: The proof is similar to the proof of the theorem using perturbation theory and the fact that eigenvalues cannot be 'sprung' from the disc of radius θ to the unit circle.

5. Abelian cocycles

As usual we shall suppose σ, τ generate a permutative mixing $\mathbb{Z}^{2+}$ action. In this section we shall be interested in abelian cocycles or, in other words, cocycles with values in the unit circle $K\ (= U(1))$. We shall only consider F_θ cocycles.

In this case cocycles form an abelian group under pointwise multiplication: $(f, g) \times (f', g') = (ff', gg')$. And cocycles cohomologous to constants form a subgroup. The resulting quotient group we call the (first) essential cohomology group. It is denoted by H_0^1 and is a topological group with the topology provided by $F_\theta(K) \times F_\theta(K)$.

In the following we use Livsic's well-known result that if $f \in F_\theta(K)$ and $f = h \circ \sigma / h$ where $h\colon X \to K$ is, say, continuous, then $h \in F_\theta(K)$.

COROLLARY 5.1: *For each* $0 < \theta < 1, H_0^1 = H_0^1(\theta)$ *is countable. If* $\theta < \theta' < 1$, *then the inclusion map (of essential cohomology classes) is an isomorphism of* $H_0^1(\theta)$ *into* $H_0^1(\theta')$.

Proof: As we have seen, the $F_\theta(K)$ cocycles, which we denote by $Z(\theta)$, form a closed subset of $F_\theta(K) \times F_\theta(K)$ and, with respect to the relative topology, each (essential) cohomology class is an open and closed set. Although we cannot claim that $Z(\theta)$ is separable in $F_\theta(K) \times F_\theta(K)$ it *is* a separable subset of $F_{\theta'}(K) \times F_{\theta'}(K)$, since F_θ functions can be $F_{\theta'}$ approximated by locally constant ('rational' valued) functions. Let S be a countable subset of $Z(\theta)$ whose $F_{\theta'}(K) \times F_{\theta'}(K)$ closure contains $Z(\theta)$. Evidently each $Z(\theta')$ (essential) cohomology class $[(f, g)]_{\theta'}$ with a representative $(f, g) \in Z(\theta)$ contains some member of S. Thus $\{[(f, g)]_{\theta'}\colon (f, g) \in Z(\theta)\}$ is countable. The proof is completed by showing that the inclusion map $[(f, g)]_\theta \to [(f, g)]_{\theta'}$ is injective. In fact if

$[(f',g')]_\theta \subset [(f,g)]_{\theta'}$, where $(f,g),(f',g') \in Z(\theta)$, then (f,g) is $F_{\theta'}$ cohomologous to (f',g') (up to a constant). It follows from Livsic that (f,g) and (f',g') are F_θ cohomologous (up to a constant). In other words $[(f,g)]_\theta = [(f',g')]_\theta$, so the inclusion map is injective.

Problem: Is H_0^1 finitely generated?

If (f,g) is locally constant then f,g are functions of a finite number of coordinates. Let $f(x) = f(x_0,\dots,x_n)$; then for each $\ell > n, L_f\colon V_\ell \to V_{\ell-1}, L_f\colon V_n \to V_n$ where V_ℓ is the finite-dimensional space of complex-valued functions which depend on only ℓ coordinates. In this case $f,g \in F_\theta(K)$ for all $\theta > 0$. Hence if the spectral radius of L_f is positive, we see that (f,g) is cohomologous to a constant. The alternative is that L_f is nilpotent on each space $V_\ell, \ell > n$.

Also, with the hypothesis that (f,g) is locally constant we have

COROLLARY 5.2: *If $f\colon \mathrm{Fix}_\sigma \to 1$ (assuming $\mathrm{Fix}_\sigma = \{x\colon \sigma x = x\} \neq \emptyset$) then (f,g) is cohomologous to a constant.*

Proof: To see this we remark that with an obvious recoding of the space X we can take f to be a function of two variables, in which case $L_f\colon V_1 \to V_1$, and takes the matrix form

$$\begin{pmatrix} f(1,1) & \dots & f(1,k) \\ \dots & \dots & \dots \\ f(k,1) & \dots & f(k,k) \end{pmatrix}$$

(zeros are to be substituted at each (i,j) where i,j is not allowed.)

Since $f\colon \mathrm{Fix}_\sigma \to 1$, the diagonal consists entirely of 1's and 0's and there is at least one 1 since $\mathrm{Fix}_\sigma \neq \emptyset$. Hence $\mathrm{Trace} L_f \neq 0$ so there is a non-zero eigenvalue. This implies that (f,g) is cohomologous to a constant.

Again with the locally constant hypothesis we have

COROLLARY 5.3: *If $f\colon \mathrm{Fix}_{\sigma^n} \to 1$ (for some $n > 0$) then (f,g) is cohomologous to a constant.*

Proof: By the last corollary applied to σ^n, τ (which is still permutative since $\sigma^n x = \sigma^n y, \tau x = \tau y$ implies $\sigma^{n-1}x = \sigma^{n-1}y$ implies $\cdots x = y$) we have (f^n, g) is (σ^n, τ) cohomologous to a constant where $f^n = f\sigma^{n-1}\cdots f\sigma f$. Hence there exists h and constants α, β such that

$$f^n = \alpha h \sigma^n h^{-1} \quad \text{and} \quad g = \beta h \tau h^{-1}.$$

The fact that (f,g) is a cocycle now implies that (f,g) is cohomologous to a constant.

COROLLARY 5.4: *The map φ which sends (cocycles modulo constants) $H_0^1(K)$ into* $\mathrm{Hom}(\mathrm{Fix}_{\sigma^n}, K)$ *(functions on the finite set* Fix_{σ^n} *with values in* K*) defined by* $\varphi(f,g) = (f^n, g)|\mathrm{Fix}_{\sigma^n}$ *is an isomorphism onto its range.*

Proof: It is clear that φ is a homomorphism. Suppose $\varphi(f,g)$ is the identity of $\mathrm{Hom}(\mathrm{Fix}_{\sigma^n} K)$, then $f^n(x) = 1$ and $g(x) = 1$ for all $x \in \mathrm{Fix}_{\sigma^n}$ so, by Corollary 5.3, (f,g) is cohomologous to a constant.

If we replace K by a finite subgroup G, we see that

THEOREM 5.5: *The group $H_0^1(G)$ of cocycles with values in the finite subgroup G of K modulo essential coboundaries is finite.*

One should note here that if $f: X \to G$ and $f = h^{-1}\sigma\alpha h$ where $h: X \to K$, then one can assume without loss of generality that $h: X \to G$, since a character annihilating G will annihilate α.

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